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成果の概要／細江陽平

京都大学教育研究振興財団より平成２３年度国際研究集会発表助成（II 期）として渡航費に対する助成を受け，２０１１年８月２８日から９月２日の日程でイタリア共和国ロンバルディア州ミラノ市において開催された the 18th IFAC World Congress （第１８回国際自動制御連盟世界会議）に参加した．IFAC World Congress は，３年に１度開催されるシステム制御分野における最も大規模かつ権威のある国際会議である．１９６０年の第１回会議がモスクワ（旧ソ連）で開催されたのをはじめとして，開催都市（国）は多岐にわたっており（１９８１年には京都でも開催された），世界中から集まったシステム制御分野の研究者・技術者がそれぞれの最新の研究成果を互いに発表しあう貴重な場の１つとなっている．第１８回となる本国際会議は，会期中２５のオーラルセッションと１つのポスターセッションが同時に進行する形で運営され，２４００を超える数の研究発表があった．筆者は主としてロバスト制御，最適制御，線形行列不等式等をキーワードとした研究に興味を持ち，時間の許す限りさまざまな研究発表を聴講した．その中にはこれまでの研究生活において扱ったことのない概念や手法に関するものも多くあり，新たな知識を得るとともに今後の研究の新たな方向性を考える貴重な機会となった．

また，筆者は８月３０日の Robustness Analysis （ロバスト性解析）というオーラルセッションにて，Relationship between Noncausal Linear Periodically Time-Varying Scaling and Causal Linear Time-Invariant Scaling for Discrete-Time Systems （離散時間系に対する非因果的周期時変スケーリングと因果的時不変スケーリングの関係）と題して研究成果の発表を行った．以下にその概要を述べる．

現実の制御対象を何らかの制御理論に基づいて制御しようと試みる際，まずはその対象をモデル化するのが一般的であるが，このときモデル化誤差（すなわち不確かさ）が生じることは避けられない．不確かさが実際に生じてしまったとしても所望の性能を保証するロバスト性という考え方は，制御の実応用上の観点から極めて重要となっている．本研究は，ロバスト性，とくにロバスト安定性の解析に関するものである．近年，解析において非因果性を導入した非因果的周期時変スケーリングという手法が提案された．この手法は従来手法である因果的時不変スケーリングに比して容易に解析結果を改善できる可能性を秘めており，筆者らはその手法がロバスト設計応用においても有効であることをこれまでの研究で示した．しかしながら，その手法と従来手法との対応関係は，既存の成果のみでは十分に議論されておらず，したがって非因果的周期時変スケーリングの優位性に理論的な保証を与えることはこれまで容易でなかった．本研究では，それら２つの手法のそれぞれにおいてある特別なものを考えると，その間にある種の等価関

係が成り立つことを初めて示した．この成果は２つの手法の間の関係を容易に考察することを可能とするものであり，本研究の今後の発展においてその中核を担うものであると確信している．

上記発表は当該セッションにおける最後の発表であり，かつその１つ前の発表が諸事情によりキャンセルとなったため，あまり多くの方に筆者の発表を聞いていただけないのではないかと心配したが，幸いにも多くの聴講者に恵まれ，本研究を広くアピールすることができた．質疑応答では海外の研究者からリフティングを介す／介さない枠組で定義されたセパレータのクラスの包含関係に関する質問を受け，その場で可能な限り回答したのち，セッション終了後により詳細な議論を行った．また，コーヒーブレイクや昼食の時間においても，日本から来られた研究者の方々から私の携わっている研究に関して説明を求められ，基礎的な事項について述べさせていただいた．本研究のキーアイディアの１つである非因果的周期時変スケーリングは，近年提案されたばかりであり，国内外のいずれにおいても認知度が高いとは言い難い状況にある．そのような中で，本国際会議のようなハイレベルかつ大規模な発表の場において多くの研究者に本研究をアピールできたことは，筆者の今後の研究生活において大変意義深いことであったと考える．今後も自身の研究を掘り下げると同時に制御工学に関する知識を広く身に付け，国際的に活躍できる研究者となれるよう精進していく所存である．

最後に，筆者の本国際会議への出席に対しご支援くださった京都大学教育研究振興財団に深く謝意を表す．

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Relationship between Noncausal Linear Periodically Time-Varying Scaling and Causal Linear Time-Invariant Scaling for Discrete-Time Systems

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Abstract: In this paper, we discuss the relationship between noncausal linear periodically time-varying (LPTV) scaling and causal linear time-invariant (LTI) scaling against discrete-time LTI closed-loop system. Noncausal LPTV scaling is naturally introduced via lifting technique, and can induce some frequency-dependent scaling in the lifting-free (i.e., usual) framework. However, it has not been clear what classes of noncausal LPTV scaling and causal LTI scaling have equivalent abilities in their respective frameworks. It is an important issue for sophisticating the theoretical base of noncausal LPTV scaling, and this paper studies such a relationship.

Keywords: Discrete-time time-invariant systems, Robust stability, Lifting, Dynamic separators

1. INTRODUCTION

Robustness is quite important in control because modeling errors cannot be avoided in practice. Useful frameworks for studying robustness such as the multiplier technique [Desoer and Vidyasagar (1975); Safonov (1980)] and integral quadratic constraints (IQC) [Megretski and Rantzer (1997)], have thus been developed. Both multiplier and IQC approaches ensure robust stability of the closed-loop system through the existence of an appropriate matrix satisfying some inequalities for a given class of uncertainties, and the relationship between these two approaches has also been discussed in Fu et al. (2005). The IQC approach provides unified treatment of robust stability conditions such as the small-gain and passivity theorems, as well as D-scaling, (D, G)-scaling and multiplier methods [Vidyasagar (1993); Zhou and Doyle (1998); Fan et al. (1991)] by appropriately confining the matrices in the theorem called separators. The separator-type robust stability theorem [Iwasaki and Hara (1998)] is also closely related to the IQC approach through the topological separation notion [Safonov (1980)], and is particularly useful for dealing with linear time-invariant (LTI) systems.

Robust stability of discrete-time LTI systems can be analyzed by searching for separators satisfying the inequality conditions in the separator-type robust stability theorem; such an approach to robust stability analysis is called causal LTI scaling. To achieve nonconservative robust stability analysis, however, such a search must work on all frequency-dependent (i.e., dynamic) separators without any constraint, but this is not feasible from a computational viewpoint. Thus, a tractable class of separators is introduced in practice, and the search of separators satisfying the robust stability condition is carried out only on that class. Such a restriction of separator classes generally leads to the conservativeness in robust stability analysis with causal LTI scaling, and reducing the conservativeness is a very important study.

To this end, discrete-time noncausal linear periodically time-varying (LPTV) scaling was proposed in Hagiwara and Ohara (2010). This approach can be naturally introduced through the application of the separator-type robust stability theorem under the lifting treatment [Bittanti and Colaneri (2000, 2009)] of discrete-time systems. It has been shown that noncausal LPTV scaling, even if we confine it to being *static*, induces some frequency-dependent (i.e., *dynamic*) causal LTI scaling under the interpretation of it in the lifting-free framework. Hence, static noncausal LPTV scaling is one of less conservative analysis approaches compared with usual static causal LTI scaling, and its effectiveness has also been demonstrated by Hagiwara and Ohara (2010); Hosoe and Hagiwara (2010b) (in robust stability analysis) and Hosoe and Hagiwara (2010a,c) (in robust controller synthesis).

In spite of such a practical success, there are some unrevealed issues about a theoretical side of noncausal LPTV scaling. As one aspect of them, we tackle in this paper the issues in the correspondence relationship between noncausal LPTV and causal LTI scaling approaches. As stated in the above, even static noncausal LPTV scaling induces some dynamic causal LTI scaling in the lifting-free framework. However, preceding studies have not explicitly characterized the class of causal LTI separators that we can equivalently deal with by working instead on noncausal LPTV scaling. To reveal such a relationship is one of the most fundamental issues for sophisticating the theoretical base of noncausal LPTV scaling. This paper provides a certain form of answer to this issue, and shows the usefulness of the answer in further studies about the comparison of a variety of noncausal LPTV and causal LTI scaling approaches.

The following part of this paper is organized as follows. Section 2 states the robust stability analysis problem we deal with, and shows the brief ideas of the analysis with lifting-free and lifting-based approaches. Section 3 reviews the definition of noncausal LPTV scaling and some existing result that gives a starting point of the issues we tackle in this paper. Section 4 introduces, on the basis of the existing result, a natural class of causal LTI scaling, and derives a class of noncausal LPTV scaling that has an equivalent ability in robust stability analysis. Section 5 applies such an equivalence relationship and discusses some further aspect of the relationship between noncausal LPTV scaling and causal LTI scaling. The validity of such discussions is also demonstrated with a numerical example of robust stability analysis. In addition, it is discussed that the idea of noncausal LPTV scaling is effective even in causal LTI scaling in the lifting-free framework, particularly in relation with the ease in the treatment of the uncertainties Δ.

2. ROBUST STABILITY THEOREM AND DISCRETE-TIME LIFTING

This section first states the robust stability analysis problem that we deal with in this paper, and then reviews two approaches to robust stability analysis. One is to analyze robust stability of systems by directly applying a robust stability theorem, and the other is to analyze it by applying such a theorem via lifting treatment. Based on the idea of the latter approach, noncausal LPTV scaling [Hagiwara and Ohara (2010)] can naturally be introduced. In this paper, we discuss the relationship between noncausal LPTV scaling introduced in the lifted framework and usual frequency-dependent scaling treated in the lifting-free framework (i.e., causal LTI scaling).

2.1 Robust stability analysis problem

This paper studies the robust stability problem of the discrete-time closed-loop system Σ shown in Fig. 1 consisting of the nominal system G and the uncertainty Δ. The nominal system G is assumed to be internally stable, finite-dimensional, LTI, and represented by

$$x_{k+1} = Ax_k + Bu_k, \quad y_k = Cx_k + Du_k \tag{1}$$

where $x_k \in \mathbf{R}^n$, $u_k \in \mathbf{R}^p$, $y_k \in \mathbf{R}^p$. The uncertainty Δ is assumed to belong to some given set $\boldsymbol{\Delta}$ satisfying the following assumption.

Assumption 1. Every $\Delta \in \boldsymbol{\Delta}$ is internally stable, finite-dimensional and LTI, and $\boldsymbol{\Delta}$ is a connected set such that $0 \in \boldsymbol{\Delta}$.

The transfer matrices of G and Δ are denoted by $G(\zeta)$ and $\Delta(\zeta)$, respectively.

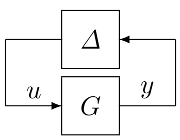

Fig. 1. Closed-loop system Σ

2.2 Separator-type robust stability theorem

To tackle the robust stability problem of Σ, we review the robust stability theorem given in Iwasaki and Hara (1998).

Theorem 1. Suppose that G is internally stable. If Σ is well-posed, $\forall \Delta \in \boldsymbol{\Delta}$, then Σ is robustly stable with respect to $\boldsymbol{\Delta}$ if and only if there exists $\Theta(\zeta) = \Theta(\zeta)^*$ ($\zeta \in \partial\mathbf{D}$) such that

$$\begin{bmatrix} I \\ G(\zeta) \end{bmatrix}^* \Theta(\zeta) \begin{bmatrix} I \\ G(\zeta) \end{bmatrix} \leq 0 \quad (\forall \zeta \in \partial\mathbf{D}) \tag{2}$$

$$\begin{bmatrix} \Delta(\zeta) \\ I \end{bmatrix}^* \Theta(\zeta) \begin{bmatrix} \Delta(\zeta) \\ I \end{bmatrix} > 0 \quad \begin{pmatrix} \forall \Delta \in \boldsymbol{\Delta}, \\ \forall \zeta \in \partial\mathbf{D} \end{pmatrix} \tag{3}$$

where $\partial\mathbf{D} := \{\zeta \in \mathbf{C} : |\zeta| = 1\}$.

See, e.g., Zhou and Doyle (1998) for the definition of well-posedness in the above theorem. The Hermitian matrix $\Theta(\zeta)$ in (2) and (3) is called a separator. We can study the robust stability of the closed-loop system Σ by searching for separators $\Theta(\zeta)$ satisfying (2) and (3) against the given $\boldsymbol{\Delta}$. We call this approach causal LTI scaling. In the next subsection, we show an alternative approach to robust stability analysis through the lifting technique.

2.3 Robust stability theorem with lifting treatment

In this subsection, we consider applying the separator-type robust stability theorem to the closed-loop system Σ via lifting treatment. We first give a brief review of the lifting technique [Bittanti and Colaneri (2000, 2009)].

The operation of constructing new signal representations $\widehat{u}_\kappa := \left[u_{\kappa N}^T, u_{\kappa N+1}^T, \cdots, u_{\kappa N+N-1}^T\right]^T$ and $\widehat{y}_\kappa := \left[y_{\kappa N}^T, y_{\kappa N+1}^T, \cdots, \ y_{\kappa N+N-1}^T\right]^T$ from the discrete-time signals u_k and y_k is called lifting of signals, where $N \geq 2$ is a positive integer. This converts the treatment of systems with input u_k and output y_k into that of systems with lifted input $\widehat{u}_\kappa$ and lifted output $\widehat{y}_\kappa$, and such treatment is called lifting of systems. The resulting lifted representations of systems are called N-lifted systems. The lifting technique is often applied to periodic systems through choosing N to be equal to the period of the systems. However, it is obvious that we can also apply the lifting technique to the LTI systems G and Δ with any $N \geq 2$. By defining $\widehat{x}_\kappa := x_{\kappa N}$, we describe the N-lifted nominal system $\widehat{G}$ by

$$\widehat{x}_{\kappa+1} = \widehat{A}\widehat{x}_\kappa + \widehat{B}\widehat{u}_\kappa, \quad \widehat{y}_\kappa = \widehat{C}\widehat{x}_\kappa + \widehat{D}\widehat{u}_\kappa. \tag{4}$$

All the coefficient matrices of $\widehat{G}$ can be constructed with the coefficient matrices in (1). The lifted nominal system $\widehat{G}$ is also LTI, and we denote its transfer matrix by $\widehat{G}(z)$; it is called the N-lifted transfer matrix of G. We can also obtain the N-lifted representation $\widehat{\Delta}$ and N-lifted transfer matrix $\widehat{\Delta}(z)$ from Δ. Through these ideas, we can obtain the lifted representation $\widehat{\Sigma}$ (Fig. 2) from the closed-loop system Σ. We remark that ζ has been used for the symbol for the z-transform of the original discrete-time signals, while z is used for that of lifted signals.

It is known from the property of lifting that Σ is robustly stable if and only if $\widehat{\Sigma}$ is. In this subsection, we consider analyzing the robust stability of the original LTI system

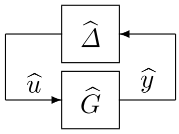

Fig. 2. Lifted closed-loop system $\widehat{\Sigma}$

Σ via analyzing that of $\widehat{\Sigma}$ through the following theorem, which follows immediately from Theorem 1.

Theorem 2. Suppose that G is internally stable. If Σ is well-posed, $\forall \Delta \in \boldsymbol{\Delta}$, then Σ is robustly stable with respect to $\boldsymbol{\Delta}$ if and only if there exists $\widehat{\Theta}(z) = \widehat{\Theta}(z)^*$ $(z \in \partial\mathbf{D})$ such that

$$\begin{bmatrix} I \\ \widehat{G}(z) \end{bmatrix}^* \widehat{\Theta}(z) \begin{bmatrix} I \\ \widehat{G}(z) \end{bmatrix} \leq 0 \quad (\forall z \in \partial\mathbf{D}) \tag{5}$$

$$\begin{bmatrix} \widehat{\Delta}(z) \\ I \end{bmatrix}^* \widehat{\Theta}(z) \begin{bmatrix} \widehat{\Delta}(z) \\ I \end{bmatrix} > 0 \quad \begin{pmatrix} \forall \Delta \in \boldsymbol{\Delta}, \\ \forall z \in \partial\mathbf{D} \end{pmatrix} \tag{6}$$

As in the previous subsection, we can analyze the robust stability of Σ by searching for separators $\widehat{\Theta}(z)$ satisfying (5) and (6) against the given $\boldsymbol{\Delta}$.

The above theorem leads to an alternative approach to the robust stability analysis of Σ. On the basis of this approach, we can naturally introduce noncausal LPTV scaling. In the next section, we review the definition of noncausal LPTV scaling and an existing result about the relationship between noncausal LPTV scaling and conventional causal LTI scaling.

3. DEFINITION OF NONCAUSAL LPTV SCALING AND REVIEW OF EXISTING RESULT

In this section, we review the definition of noncausal LPTV scaling, and the fact shown in Hagiwara and Ohara (2010) that noncausal LPTV scaling induces some class of dynamic causal LTI scaling in the lifting-free framework.

3.1 Definition of noncausal LPTV scaling

The separators $\widehat{\Theta}(z)$ in Theorem 2 have been classified into two types (causal or noncausal type) from the causality viewpoint. The noncausal type is more general and thus effective than the causal type, and the present paper is primarily concerned with the noncausal type. To make clearer the feature of the robust stability analysis with the noncausal type, however, we begin by reviewing the definitions of both types of separators, as well as the associated two types of LPTV scaling. First, causal LPTV separators are defined as follows [Hagiwara and Ohara (2010)].

Definition 1. A separator given by

$$\widehat{\Theta}(z) = \left[\widehat{V}_1(z)\ \widehat{V}_2(z)\right]^* \widehat{\Lambda} \left[\widehat{V}_1(z)\ \widehat{V}_2(z)\right] \tag{7}$$

is called a causal LPTV separator, where $\widehat{V}_1(z)$ and $\widehat{V}_2(z)$ are the N-lifted transfer matrices of causal N-periodic systems V_1 and V_2 with p inputs, respectively, and $\widehat{\Lambda} = \widehat{\Lambda}^*$ is a constant matrix of the form $\widehat{\Lambda} = \mathrm{diag}[\Lambda_1, \cdots, \Lambda_N]$ with the size of Λ_i being the same for all $i = 1, \cdots, N$ and compatible with V_1 and V_2. In particular, if V_1 and V_2 are static, then the corresponding separator is called a static causal LPTV separator.

The approach to robust stability analysis based on causal LPTV separators defined above is called causal LPTV scaling. Even though it has been described in the framework of lifting, such scaling corresponds to the conventional lifting-free scaling with causal LPTV systems.

On the other hand, a noncausal LPTV separator [Hagiwara and Ohara (2010)] has been defined by generalizing Definition 1.

Definition 2. A separator given by $\widehat{\Theta}(z) = \widehat{V}(z)^* \Gamma \widehat{V}(z)$ is called a noncausal LPTV separator, where $\widehat{V}(z)$ is the transfer matrix of a causal LTI system $\widehat{V}$ with $2Np$ inputs *defined on the lifted time axis* and $\Gamma = \Gamma^*$ is a constant matrix of compatible size. In particular, if $\widehat{V}$ is static, then the corresponding separator is called a static noncausal LPTV separator.

The approach to robust stability analysis based on noncausal LPTV separators defined above is called noncausal LPTV scaling.

Comparing Definition 2 with Definition 1, we readily see that the structure of noncausal LPTV separators is generalized from that of causal LPTV separators. Since noncausal LPTV scaling is more flexible than causal LPTV scaling in this way, the former is known to be generally more effective for robust stability analysis than the latter. In this section, we review an existing result about the relationship between noncausal LPTV scaling with lifting treatment and usual scaling without lifting treatment.

3.2 Causal LTI scaling induced by noncausal LPTV scaling

In the previous subsection, we reviewed the definitions of causal LPTV scaling and its generalized form called noncausal LPTV scaling. In this subsection, we further review an existing result showing that noncausal LPTV scaling with lifting treatment induces some dynamic causal LTI scaling in the lifting-free framework even if the noncausal LPTV scaling is static. More precisely, we have the following theorem [Hagiwara and Ohara (2010)].

Theorem 3. If a separator $\widehat{\Theta}(z) = \widehat{\Theta}_0(z)$ satisfies (5) and (6) in the N-lifted framework, then the separator

$$\Theta(\zeta) = \mathcal{T}(\zeta)^* \widehat{\Theta}_0(\zeta^N) \mathcal{T}(\zeta) \tag{8}$$

satisfies (2) and (3) in the lifting-free framework, where

$$\mathcal{T}(\zeta) := \mathrm{diag}[T_p(\zeta), T_p(\zeta)],\ T_p(\zeta) := \begin{bmatrix} \zeta^{-(N-1)} I_p \\ \vdots \\ \zeta^{-1} I_p \\ I_p \end{bmatrix}. \tag{9}$$

This theorem implies that if we obtain $\widehat{\Theta}(z) = \widehat{\Theta}_0(z)$ satisfying (5) and (6) in the lifted framework, then we immediately obtain $\Theta(\zeta)$ satisfying (2) and (3). In particular, since Theorem 3 is true even if $\widehat{\Theta}_0(z)$ is constrained to a static one (i.e., constant matrix $\widehat{\Theta}_0$), even static noncausal LPTV scaling induces some frequency-dependent scaling (i.e., dynamic causal LTI scaling) if it is interpreted in the lifting-free framework. This is one of the advantages of noncausal LPTV scaling, and demonstrated in Hagiwara and Ohara (2010) and Hosoe and Hagiwara (2010b) to lead to effectiveness for reducing the conservativeness of robust stability analysis, and also for robust controller synthesis in Hosoe and Hagiwara (2010a,c).

We have reviewed in the above an existing result clarifying to some extent the relationship between noncausal LPTV scaling and causal LTI scaling. However, having only such a result is not yet enough as a study of the correspondence relationship between these two types of scaling approaches. Next section states detailed reasons why the above result alone is not enough as a comparison of these scaling approaches, and provides a further result about their mutual relationship.

4. RELATIONSHIP BETWEEN NONCAUSAL LPTV SCALING AND CAUSAL LTI SCALING

In this section, we first describe the issue to be clarified further about the relationship between noncausal LPTV scaling and causal LTI scaling. We then provide a certain result for such an issue. More specifically, we derive as one of the main results of this paper some class of *dynamic* noncausal LPTV separators that has an ability equivalent to a natural class (with a simple parametrization) of causal LTI separators, where the latter contains the class of separators induced on the lifting-free framework by the class $\widehat{\boldsymbol{\Theta}}_{\mathrm{static}}$ of *static* noncausal LPTV separators. A theoretical implication of such a result is also discussed.

4.1 Overview of the relationship given by existing result

In this paper, we study the relationship between noncausal LPTV scaling and causal LTI scaling, and from such a point of view, we reviewed Theorem 3 in the previous section giving a partial result on the relationship. It shows that noncausal LPTV scaling with lifting treatment induces some dynamic causal LTI scaling when it is interpreted in the lifting-free framework, even if the noncausal LPTV scaling is static. However, it is not generally clear if the inverse assertion of Theorem 3 is also true. That is, it is not clear if a static $\widehat{\Theta}(z) = \widehat{\Theta}_0$ satisfies (5) and (6) whenever it together with (8) leads to $\Theta(\zeta)$ satisfying (2) and (3). More precisely speaking, when we define

$$\boldsymbol{\Theta}(\zeta) := \{\Theta(\zeta) \text{ given by (8)} \mid \widehat{\Theta}_0 \in \widehat{\boldsymbol{\Theta}}_{\mathrm{static}}\} \qquad (10)$$

(which depends on N and corresponds to a subset of all causal LTI separators), it is not clear whether we can reduce the problem of searching for $\Theta(\zeta) \in \boldsymbol{\Theta}(\zeta)$ equivalently into the problem of searching for static noncausal LPTV separators $\widehat{\Theta}(z) = \widehat{\Theta}_0$. We give a schematic picture of the above relationship in Fig. 3, where the circles denote the classes of separators we consider here for lifting-free and lifted frameworks. If we could succeed in revealing in more details the correspondence relationships between both frameworks (e.g., what class of causal LTI scaling corresponds to some class of noncausal LPTV scaling and vice versa), we would be able to understand more deeply about the advantages and drawbacks of different types of scaling approaches. Furthermore, making the relationships clear is quite an important issue also for further development of the theory of noncausal LPTV scaling. Aiming at taking a step forward to this issue, this paper introduces a class of dynamic noncausal LPTV separators denoted by $\widehat{\boldsymbol{\Theta}}(z)$, and shows that it has an equivalent ability to what

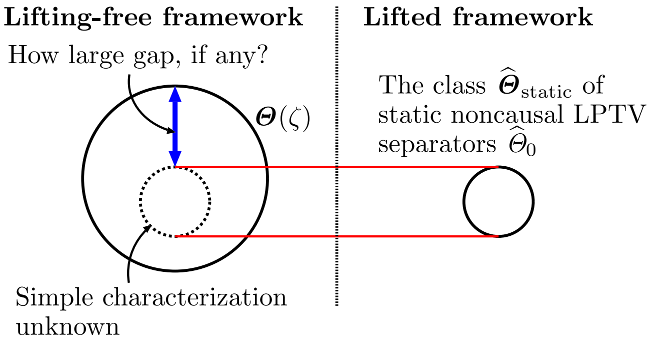

Fig. 3. Schematic picture of the correspondence relationships among different classes of separators

the class of causal LTI separators $\boldsymbol{\Theta}(\zeta)$ possesses in the lifting-free framework. We also discuss the properties of the scaling with such $\widehat{\boldsymbol{\Theta}}(z)$.

4.2 Equivalence relationship between some classes of noncausal LPTV scaling and causal LTI scaling

As we have seen in the preceding discussions, even *static* noncausal LPTV scaling generally induces dynamic (and thus frequency-dependent) causal LTI scaling if it is interpreted in the lifting-free framework. This might suggest the use of static noncausal LPTV scaling as an alternative tool for searching for dynamic scaling in the lifting-free framework, and such an approach is indeed favorable due to the ease in the search of *static* separators. However, even though the static noncausal LPTV separator $\widehat{\Theta}_0$ induces $\Theta(\zeta) \in \boldsymbol{\Theta}(\zeta)$ as seen from Theorem 3 and (10), it is not clear if such an approach is equivalent (i.e., not conservative) to the conventional lifting-free treatment with the separator $\Theta(\zeta)$ considered in the whole class $\boldsymbol{\Theta}(\zeta)$; see Fig. 3. Stimulated by this question, we introduce in this subsection a class of noncausal LPTV separators denoted by $\widehat{\boldsymbol{\Theta}}(z)$, and show that noncausal LPTV scaling with such a separator class is the one that has an equivalent ability to the lifting-free LTI scaling with the separator class $\boldsymbol{\Theta}(\zeta)$. The following theorem plays a crucial role for the construction of such $\widehat{\boldsymbol{\Theta}}(z)$; the proof is omitted because of limited space.

Theorem 4. Given a Hermitian matrix $\widehat{\Theta}_0$, the separator $\Theta(\zeta)$ given by (8) satisfies (2) and (3) if and only if the separator given by

$$\widehat{\Theta}(z) = \frac{1}{N}\sum_{l=0}^{N-1} (\mathcal{S}^l)^* \widehat{\Theta}_0 \mathcal{S}^l \qquad (11)$$

satisfies (5) and (6) in the N-lifted framework, where $\mathcal{S}$ is the z-dependent matrix given by

$$\mathcal{S} := \mathrm{diag}[S_p, S_p], \quad S_p := S_p(z) = \begin{bmatrix} 0 & z^{-1}I_p \\ I_{(N-1)p} & 0 \end{bmatrix}. \qquad (12)$$

In view of (11), we define $\widehat{\boldsymbol{\Theta}}(z)$ as follows.

$$\widehat{\boldsymbol{\Theta}}(z) := \{\widehat{\Theta}(z) \text{ given by (11)} \mid \widehat{\Theta}_0 \in \widehat{\boldsymbol{\Theta}}_{\mathrm{static}}\}. \qquad (13)$$

Then, the aforementioned "equivalence" between $\boldsymbol{\Theta}(\zeta)$ and $\widehat{\boldsymbol{\Theta}}(z)$ follows immediately. A schematic picture of the situation is given in Fig. 4. We remark that this figure, unlike Fig. 3, does not represent the classes of separators but represents (the level of) the abilities of different types

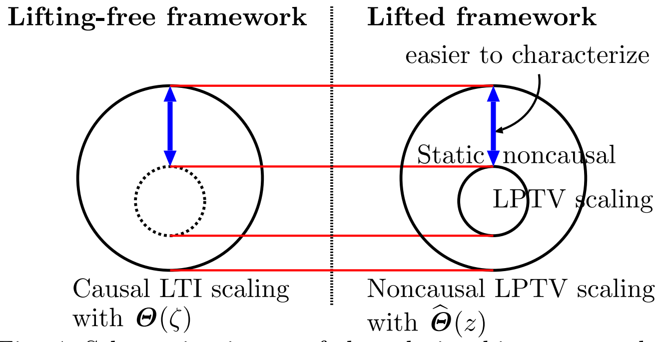

Fig. 4. Schematic picture of the relationships among the abilities of different types of scaling approaches

of scaling approaches with the sizes of the circles; note that the class $\widehat{\boldsymbol{\Theta}}_{\mathrm{static}}$ of static noncausal LPTV separators is *not* a subset of $\widehat{\boldsymbol{\Theta}}(z)$ but the ability of the former does not exceed that of the latter, as indicated in the figure. Let us further observe a relationship between the classes $\widehat{\boldsymbol{\Theta}}(z)$ and $\widehat{\boldsymbol{\Theta}}_{\mathrm{static}}$. As a direct consequence of Theorems 3 and 4, we can see that if a static noncausal LPTV separator $\widehat{\Theta}(z) = \widehat{\Theta}_0$ satisfies (5) and (6), then $\widehat{\Theta}(z) \in \widehat{\boldsymbol{\Theta}}(z)$ given by (11) with the same $\widehat{\Theta}_0$ also satisfies (5) and (6) again. However, it should be noted that it is not clear, in general, if the converse is true. That is, even if a $\widehat{\Theta}(z) \in \widehat{\boldsymbol{\Theta}}(z)$ satisfies (5) and (6), it is not clear if the same inequalities hold also with any (or even a suitably chosen) static noncausal LPTV separator $\widehat{\Theta}_0$ leading to that $\widehat{\Theta}(z)$ through (8). This corresponds to the possible gap between the scaling with the separator class $\widehat{\boldsymbol{\Theta}}(z)$ and that with the class $\widehat{\boldsymbol{\Theta}}_{\mathrm{static}}$, as shown in Fig. 4 with respect to the lifted framework. This gap corresponds to that (i.e., our initial concern in the lifting-free framework) in Fig. 3, where the former gap is expected to be more tractable than the latter because each of the corresponding classes of separators has been characterized explicitly in the lifted framework.

This consequence gives a partial answer to the unresolved issues left behind Theorem 3 about the properties and ability of static noncausal LPTV scaling. The significance of Theorem 4 lies in leading to such a consequence, with which we can advance the study about the theoretical aspects of noncausal LPTV scaling in the following section, and hopefully in more details in the future.

Remark 1. The sufficiency assertion of Theorem 4 could be proved through direct but tedious computations by applying Theorem 3, i.e., by substituting $\widehat{\Theta}(z)$ given by (11) into $\widehat{\Theta}_0(z)$ in (8). The necessity assertion, however, does not follow from Theorem 3, and thus this assertion corresponds to the theoretical advance that Theorem 4 has made over Theorem 3, as initially intended. Our omitted proof is based on the properties of the transfer matrices of N-lifted systems and the Nth root of 1, and reversing the arguments leads to the sufficiency proof.

Remark 2. The assertion of Theorem 4 remains valid even if $\widehat{\Theta}_0$ is replaced by a dynamic separator $\widehat{\Theta}_0(z)$. However, simply because the interest of this paper lies mostly in the class $\boldsymbol{\Theta}(\zeta)$ introduced in (10), $\widehat{\Theta}_0$ has been assumed to be static accordingly.

5. FURTHER DISCUSSION ABOUT RELATIONSHIP BETWEEN THE TWO SCALING APPROACHES

On the basis of Theorem 4, we have introduced in the previous section the class $\widehat{\boldsymbol{\Theta}}(z)$ of noncausal LPTV separators that has a completely equivalent ability to the class $\boldsymbol{\Theta}(\zeta)$ of causal LTI separators given by (8) with $\widehat{\Theta}_0 \in \widehat{\boldsymbol{\Theta}}_{\mathrm{static}}$. This section first shows a structural property of the noncausal LPTV separators $\widehat{\Theta}(z) \in \widehat{\boldsymbol{\Theta}}(z)$, and then further discusses the relationship between noncausal LPTV scaling and causal LTI scaling. In particular, we introduce some new classes of separators other than $\boldsymbol{\Theta}(\zeta)$ and $\widehat{\boldsymbol{\Theta}}(z)$ both in the lifting-free and lifted frameworks, and demonstrate the importance of Theorem 4 in revealing the mutual relationships among the scaling approaches with such separator classes.

5.1 Interpretation of the separator class $\widehat{\boldsymbol{\Theta}}(z)$ via timing shift

In Subsection 3.2, we reviewed Theorem 3 about the implication of (static) noncausal LPTV scaling interpreted in the framework of conventional lifting-free treatment. This led us to the introduction of the class $\boldsymbol{\Theta}(\zeta)$ of causal LTI separators given by (10). In addition, by considering a noncausal LPTV separator $\widehat{\Theta}(z)$ in the form of (11) instead of the static noncausal LPTV separator $\widehat{\Theta}(z) = \widehat{\Theta}_0$, we gave, as Theorem 4, a sort of converse assertion of Theorem 3. This led to the introduction of the class $\widehat{\boldsymbol{\Theta}}(z)$ of noncausal LPTV separators given in (13), as a class that is equivalent to the class $\boldsymbol{\Theta}(\zeta)$ of causal LTI separators for the lifting-free treatment. With this in mind, we study in this subsection a structural property of noncausal LPTV separators $\widehat{\Theta}(z) \in \widehat{\boldsymbol{\Theta}}(z)$.

We begin with the properties of S_p in (12). It is related with the timing shift matrix Z_p introduced in Hosoe and Hagiwara (2010b) by $S_p = Z_p^{-1}$, and satisfies

$$S_p S_p^* = S_p^* S_p = I, \quad S_p^N = z^{-1} I_{pN} \quad (z \in \partial\mathbf{D}). \tag{14}$$

We call this S_p a timing back-shift matrix. Applying the similarity transformation by S_p on the transfer matrix of the N-lifted LTI representation of an N-periodic system corresponds to lifting that N-periodic system with its input and output signals advanced by a unit discrete-time, and then introducing the transfer matrix of the resulting lifted LTI system. Hence, such a similarity transformation leads to a different lifted transfer matrix, in general. However, if it is applied on the lifted transfer matrices of LTI systems such as G and Δ in the present paper, then it induces no variations of the transfer matrices, i.e., the transfer matrices remain invariant. More precisely, $S_p^{-1}\widehat{G}(z)S_p = \widehat{G}(z)$ and $S_p^{-1}\widehat{\Delta}(z)S_p = \widehat{\Delta}(z)$.

From these relations, post-multiplying S_p and pre-multiplying $S_p^{-1} = S_p^*$ on (5) and (6) lead to the fact that (a) $\widehat{\Theta}(z) = \widehat{\Theta}_0(z)$ satisfies these inequalities if and only if $\widehat{\Theta}(z) = \mathcal{S}^*\widehat{\Theta}_0(z)\mathcal{S}$ satisfies the same inequalities. In particular, (b) $\widehat{\Theta}(z) = \widehat{\Theta}_0 \in \widehat{\boldsymbol{\Theta}}_{\mathrm{static}}$ satisfies (5) and (6) if and only if the timing-shifted separator $\widehat{\Theta}(z) = \mathcal{S}^*\widehat{\Theta}_0\mathcal{S}$ satisfies the same inequalities. Repeating this argument, we see that $\widehat{\Theta}(z) = \widehat{\Theta}_0 \in \widehat{\boldsymbol{\Theta}}_{\rm static}$ satisfies (5) and (6) if and only if $\widehat{\Theta}(z) = (\mathcal{S}^l)^* \widehat{\Theta}_0 \mathcal{S}^l$ satisfies the same inequalities for all $l = 0, 1, \cdots$. However, considering all $l = 0, 1, \cdots$ is redundant if we note from (14) that

$$(\mathcal{S}^{l+N})^* \widehat{\Theta}_0 \mathcal{S}^{l+N} = (\mathcal{S}^l)^* \widehat{\Theta}_0 \mathcal{S}^l, \quad z \in \partial\mathbf{D} \tag{15}$$

and it is enough to consider only $l = 0, \cdots, N-1$ in the above observation. We can further see that the separator $\widehat{\Theta}(z)$ given in Theorem 4 is nothing but the average of these N timing-shifted separators. In particular, it follows again from (15) that the average is such a special separator that is invariant under the congruence transformation by $\mathcal{S}$, i.e., $\mathcal{S}^* \widehat{\Theta}(z) \mathcal{S}$ remains the same as $\widehat{\Theta}(z)$. For simplicity, we say that such separators are shift-invariant. It can be interpreted that the success of Theorem 4 in providing a sort of converse assertion of Theorem 3 stems from the introduction of such a shift-invariant separator. Indeed, in Theorem 3, only $\widehat{\Theta}_0 \in \widehat{\boldsymbol{\Theta}}_{\rm static}$ has been considered in the lifted framework, but it is not shift-invariant, in general, and thus the above observation (b) leads to a new but equally effective timing-shifted separator. The price we have paid to arrive at the above "converse" (to be able to ensure the equivalence of the robust stability analysis in the lifted framework and that with the separator class $\boldsymbol{\Theta}(\zeta)$ in the lifting-free framework) is that we are led to the "$\mathcal{S}$-invariant" class $\widehat{\boldsymbol{\Theta}}(z)$ of *dynamic* noncausal LPTV separators instead of $\widehat{\boldsymbol{\Theta}}_{\rm static}$. In this regard, it would be worth noting that, in spite of the existence of $\mathcal{S}^l$, $l = 0, \cdots, N-1$ in (11), the class $\widehat{\boldsymbol{\Theta}}(z)$ consists of only such separators represented as linear combinations of z^i, $i = -1, 0, 1$. This is because $S_p(z)^l$ has the same property as can be verified from the definition of S_p (see also (14)).

5.2 Comparison of causal LTI and noncausal LPTV scaling approaches based on equivalence relationship

In the previous subsection, we briefly discussed the properties of noncausal LPTV scaling based on $\widehat{\boldsymbol{\Theta}}(z)$ from the viewpoint of timing shift introduced in Hosoe and Hagiwara (2010b). In this subsection, using the structural property of $\widehat{\Theta}(z)$ given by (11), we discuss further relationships between noncausal LPTV scaling and causal LTI scaling and clarify some relative advantages and drawbacks of the use of these approaches.

We first consider introducing some classes of separators in addition to $\boldsymbol{\Theta}(\zeta)$ and $\widehat{\boldsymbol{\Theta}}(z)$. We begin by introducing the class of the separators given by

$$\widehat{\Theta}(z) = \left(\begin{bmatrix} z^{-1} I_{Np} \\ I_{Np} \end{bmatrix}^* \widehat{\Theta}_{0,ij}^{2N} \begin{bmatrix} z^{-1} I_{Np} \\ I_{Np} \end{bmatrix} \right)_{i,j=1,2}, \tag{16}$$
$$\widehat{\Theta}_0^{2N} = (\widehat{\Theta}_{0,ij}^{2N})_{i,j=1,2} \in \widehat{\boldsymbol{\Theta}}_{\rm static}^{2N}$$

where $\widehat{\boldsymbol{\Theta}}_{\rm static}^{2N}$ is the class of static noncausal LPTV separators defined in the $2N$-lifted framework (we consider it *even though we do remain at the moment in the N-lifted framework; thus (16) is indeed a separator for the N-lifted framework as seen from its size*). Since every $\widehat{\Theta}(z) \in \widehat{\boldsymbol{\Theta}}(z)$ is represented as a linear combination of only z^i, $i = -1, 0, 1$ from the definition of $S_p(z)$, we can see that the class $\widehat{\boldsymbol{\Theta}}(z)$ is included in the class of the

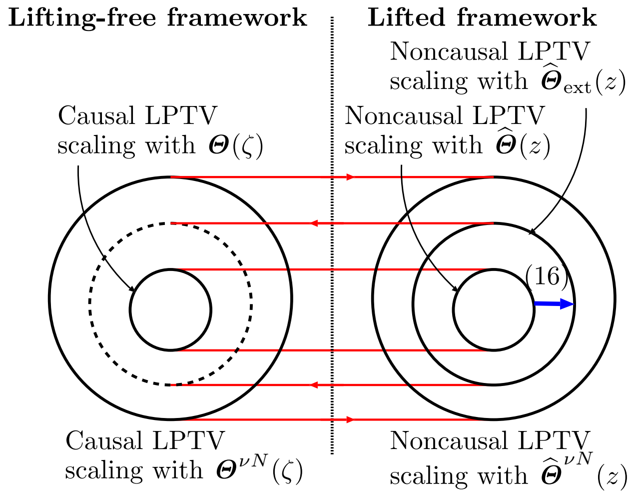

Fig. 5. Schematic picture of further relationships among the abilities of other different types of scaling approaches

separators given by (16), which we denote by $\widehat{\boldsymbol{\Theta}}_{\rm ext}(z)$. Hence, it is obvious that the robust stability analysis based on the separator class $\widehat{\boldsymbol{\Theta}}_{\rm ext}(z)$ is less (more precisely, at least no more) conservative than that with $\widehat{\boldsymbol{\Theta}}(z)$. Then, by regarding $\widehat{\Theta}(z) \in \widehat{\boldsymbol{\Theta}}_{\rm ext}(z)$ as $\widehat{\Theta}_0(z)$ in (8) and applying Theorem 3, we can see that the noncausal LPTV scaling with the separator class $\widehat{\boldsymbol{\Theta}}_{\rm ext}(z)$ also induces some class of causal LTI scaling in the lifting-free framework (see the dashed circled in Fig. 5, which is again meant to represent the abilities of different types of scaling approaches, as in Fig. 4). By direct computation of (8) resulting from the above argument, we see that such an induced class is included in the class of lifting-free causal LTI scaling with $\boldsymbol{\Theta}^{2N}(\zeta)$, which is defined by (10) with the underlying N replaced by $2N$ (this corresponds to the outermost circle in the lifting-free framework with $\nu = 2$). Moreover, by applying Theorem 4, we can obtain the class of separators $\widehat{\boldsymbol{\Theta}}^{2N}(z)$ (see the outermost circle in the lifted framework) such that the noncausal LPTV scaling with it has an equivalent ability to the causal LTI scaling with $\boldsymbol{\Theta}^{2N}(\zeta)$. Here, it should be noted that $\widehat{\boldsymbol{\Theta}}^{2N}(z)$ is a class *defined (and thus to be used) in the $2N$-lifted framework.* We remark that the above arguments are limited to the case of $\nu = 2$, for simplicity, but readily generalize to the arguments in the νN-lifted framework. This situation is also shown in Fig. 5.

To summarize, by applying Theorem 3 and 4, we can reveal the equivalences and differences in the abilities of various types of scaling approaches introduced here, in terms of the conservativeness of robust stability analysis. Aside from such a theoretical aspect, however, the computation load is also an important factor for the comparison of these scaling approaches. We next discuss their relationships from such a viewpoint, but for simplicity, we confine ourselves to the noncausal LPTV scaling with $\widehat{\boldsymbol{\Theta}}_{\rm static}$ and $\widehat{\boldsymbol{\Theta}}_{\rm ext}(z)$, and the causal LTI scaling with $\boldsymbol{\Theta}(\zeta)$ and $\boldsymbol{\Theta}^{2N}(\zeta)$. Each of the separator class above is parametrized by a constant matrix, and as stated in Appendix A, we can have an exact search method for such a constant matrix through the KYP lemma [Rantzer (1996)], provided that we conform

Table 1. Comparison of computation load

Scaling type	(i)	(ii)	(iii)
noncausal, $\widehat{\boldsymbol{\Theta}}_{\rm static}$	$2pN$	n	$pN+n$
causal, $\boldsymbol{\Theta}(\zeta)$	$2pN$	$pN+n-p$	$3pN+n-2p$
noncausal, $\widehat{\boldsymbol{\Theta}}_{\rm ext}(z)$	$4pN$	$pN+n$	$4pN+n$
causal, $\boldsymbol{\Theta}^{2N}(\zeta)$	$4pN$	$2pN+n-p$	$6pN+n-2p$

to the standard technique in robust stability analysis with respect to the treatment of the inequalities (3) and (6) about the uncertainties Δ. Such a technique leads to LMI conditions with which the parameter matrix in each of the classes $\widehat{\boldsymbol{\Theta}}_{\rm static}$, $\widehat{\boldsymbol{\Theta}}_{\rm ext}(z)$, $\boldsymbol{\Theta}(\zeta)$ and $\boldsymbol{\Theta}^{2N}(\zeta)$ can be searched for.

Now, there are three main factors affecting the computation load; (i) the size of $\widehat{\Theta}_0 \in \widehat{\boldsymbol{\Theta}}_{\rm static}$ (or $\widehat{\Theta}_0^{2N} \in \widehat{\boldsymbol{\Theta}}_{\rm static}^{2N}$) we essentially search for, (ii) the order of the "nominal system" $G_T(\zeta)$ that arises in the process of reduction to LMI (see Appendix A), which determines the size of the Lyapunov matrix in the resulting LMI, and (iii) the size of the resulting LMI itself. These three factors can be studied easily, and can be summarized as shown in Table 1. According to this table, we can see that a lower line has a larger value in each of (i), (ii) and (iii) (since we may reasonably assume $N > 1$) and thus a less conservative approach (recall the relation in Fig. 5) is considered to tend to take a more computation time. This suggests a natural consequence that, to obtain a sharper result in robust stability analysis with whatever scaling approach in the lifting-free or lifted framework, it basically has to be allowed for a more computation time.

In spite of the above observation, however, there clearly exists an advantage in the idea of noncausal LPTV scaling. Indeed, there is an important qualitative fact concealed behind the quantitative comparison in Table 1 but suggesting the effectiveness of the idea of noncausal LPTV scaling even in the search for the separators in the classes $\boldsymbol{\Theta}(\zeta)$ and $\boldsymbol{\Theta}^{2N}(\zeta)$ for causal LTI scaling. Such an important fact can be described as follows. In causal LTI scaling, $\Theta(\zeta)$ should satisfy (3), and thus the parameter matrix $\widehat{\Theta}_0 \in \widehat{\boldsymbol{\Theta}}_{\rm static}$ or $\widehat{\Theta}_0^{2N} \in \widehat{\boldsymbol{\Theta}}_{\rm static}^{2N}$ in $\Theta(\zeta)$ must be confined to a convex set such that (3) is satisfied regardless of $\Delta \in \boldsymbol{\Delta}$; otherwise, the LMI condition derived from (2) through the KYP lemma cannot be solved easily for $\widehat{\Theta}_0$ or $\widehat{\Theta}_0^{2N}$. Construction of such a convex set for the lifting-free framework is actually deeply related to the seemingly irrelevant noncausal LPTV scaling approach *in the lifted framework*, and can be carried out easily with the idea of noncausal LPTV scaling (see Appendix A for more details).

As we have seen in the above, we can explicitly compare a variety of scaling approaches in both lifting-free and lifted frameworks by making use of Theorem 4. This clearly demonstrates the significance of Theorem 4 as our main result in this paper.

5.3 Numerical example of robust stability analysis

This subsection numerically confirms the facts stated in the previous subsection, i.e., the relationship among the different types of scaling approaches in terms of their conservativeness as shown in Fig. 5, and that in terms of their computation load as discussed in Table 1.

Table 2. Comparison of the analysis results ($N = 2$)

Scaling type	$\overline{\delta}$	Computation time
noncausal, $\widehat{\boldsymbol{\Theta}}_{\rm static}$	1.2360	4.81 sec
causal, $\boldsymbol{\Theta}(\zeta)$	1.4938	6.41 sec
noncausal, $\widehat{\boldsymbol{\Theta}}_{\rm ext}(z)$	1.4942	8.21 sec
causal, $\boldsymbol{\Theta}^{2N}(\zeta)$	1.4942	13.94 sec

We consider the internally stable LTI system G given by

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{ccccc|cc} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0.1 \\ -0.2 & -0.62 & 0.01 & 0.6 & -0.7 & 0.1 & 0 \\ \hline 1 & 1 & 1 & 0 & 0 & 0.1 & 0.2 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right]. \quad (17)$$

In addition, we assume the corresponding uncertainties Δ are static LTI and structured as given by $\Delta = {\rm diag}[\delta_1\ \delta_2]$. The purpose here is to compute (a lower bound of) the maximum $\overline{\delta}$ such that the closed-loop system Σ is robustly stable against the uncertainty set $\boldsymbol{\Delta} = \{\Delta : \|\Delta\| < \overline{\delta}\}$. More precisely, we apply the four scaling approaches in Table 1 and compare the analysis results and the computation times.

The results for $N = 2$ are shown in Table 2, where we confined ourselves to the class of (D, G)-scaling type separators [Fan et al. (1991)] in each of those approaches.

Since larger resulting $\overline{\delta}$ implies that less conservative analysis has been achieved, Table 2 shows that a scaling approach in a lower row leads to less conservative robust stability analysis. We see that this is indeed consistent with the discussions in the preceding subsection (see Fig. 5). At the price for the improvement of the accuracy in the analysis, we also see that the computation times grow. We see that this is also consistent with the observation in Table 1.

6. CONCLUSION

In this paper, we aimed at revealing some aspect of the relationship between noncausal LPTV scaling in the lifted framework and causal LTI scaling in the lifting-free framework. On the basis of an existing result showing that even static noncausal LPTV scaling generally induces dynamic (i.e., frequency-dependent) scaling in the lifting-free framework, we first introduced the class $\boldsymbol{\Theta}(\zeta)$ of causal LTI separators in the lifting-free framework such that the induced equivalent dynamic scaling can be related to a separator $\Theta(\zeta) \in \boldsymbol{\Theta}(\zeta)$. We then introduced the class $\widehat{\boldsymbol{\Theta}}(z)$ of noncausal LPTV separators in the lifted framework and proved that noncausal LPTV scaling with $\widehat{\boldsymbol{\Theta}}(z)$ has an equivalent ability to causal LTI scaling with $\boldsymbol{\Theta}(\zeta)$ in robust stability analysis. Moreover, based on such a newly derived equivalence relationship, we explicitly compared some classes of causal LTI and noncausal LPTV scaling approaches in terms of the conservativeness of robust stability analysis and the computation load. The validity of such discussions has been confirmed by a numerical example. In addition, we also discussed an important issue about the effectiveness of the idea of noncausal LPTV scaling even in the lifting-free framework with causal LTI separators, particularly in relation with the ease in the treatment of the uncertainties Δ and the construction of an appropriate separator class in such a framework.

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Appendix A. A KEY IDEA FOR THE SEARCH OF DYNAMIC SEPARATORS SATISFYING ROBUST STABILITY CONDITIONS VIA KYP LEMMA

Suppose that we are to search for $\widehat{\Theta}_0 \in \widehat{\boldsymbol{\Theta}}_{\rm static}$ such that $\Theta(\zeta)$ given by (8) satisfies (2) and (3). Substituting (8) into (2) leads to

$$\begin{bmatrix} I_p \\ G_T(\zeta) \end{bmatrix}^* \widehat{\Theta}_0' \begin{bmatrix} I_p \\ G_T(\zeta) \end{bmatrix} \le 0, \quad G_T(\zeta) = \begin{bmatrix} \zeta^{-1} I_p \\ \vdots \\ \zeta^{-(N-1)} I_p \\ G(\zeta) \\ \vdots \\ \zeta^{-(N-1)} G(\zeta) \end{bmatrix} \tag{A.1}$$

by introducing the permutation matrix P such that $[T_p(\zeta)^T, (T_p(\zeta)G(\zeta))^T]^T = P[I_p, G_T(\zeta)^T]^T$, where $\widehat{\Theta}_0' = P^T \widehat{\Theta}_0 P$. Hence, we can regard the problem stated above as that of searching for static $\widehat{\Theta}_0'$ for $G_T(\zeta)$ viewed as the nominal system. Such a problem can be solved exactly through the well-known KYP lemma [Rantzer (1996)] as long as the other constraint (3) on $\widehat{\Theta}_0'$ (or $\widehat{\Theta}_0$) is handled properly. Conforming to the standard technique in robust stability analysis, it is reasonable for us to restrict $\widehat{\Theta}_0$ to a subset $\widehat{\boldsymbol{\Theta}}_{{\rm static},\boldsymbol{\Delta}}$ of $\widehat{\boldsymbol{\Theta}}_{\rm static}$, where such a subset is required to have the property that every $\widehat{\Theta}_0 \in \widehat{\boldsymbol{\Theta}}_{{\rm static},\boldsymbol{\Delta}}$ together with (8) yields $\Theta(\zeta)$ that satisfies (3) for any $\Delta \in \boldsymbol{\Delta}$.

Construction of such a subset might look an intricate problem particularly because we must deal with the dynamic separator $\Theta(\zeta)$ and thus (3) does not reduce to a static inequality even if we were to consider only static uncertainties Δ. Nevertheless, such a subset can actually be constructed rather easily, and Theorem 3 plays a significant role in the construction; it follows immediately from this theorem that we may take $\widehat{\boldsymbol{\Theta}}_{{\rm static},\boldsymbol{\Delta}}$ to be a subset of $\widehat{\boldsymbol{\Theta}}_{\rm static}$ such that $\widehat{\Theta}(z) = \widehat{\Theta}_0 \in \widehat{\boldsymbol{\Theta}}_{{\rm static},\boldsymbol{\Delta}}$ satisfies (6). Such a subset has been introduced in Hosoe and Hagiwara (2010a) and Hosoe and Hagiwara (2010c) in an explicit form, in accordance with the block diagonal structure of Δ consisting not only of static but also dynamic subblocks. The construction of such a class viewed in the lifted framework is in fact fairly easy as an extension of similar techniques in μ-analysis [Zhou and Doyle (1998)]. This implies an important fact that the idea of noncausal LPTV scaling is quite useful even if we were to carry out robust stability analysis within the lifting-free framework through the separator class $\boldsymbol{\Theta}(\zeta)$, particularly with respect to the treatment of the uncertainties Δ and thus the inequality (3).

It is obvious that we can also search for dynamic separators $\widehat{\Theta}(z)$ given by (11) or (16) (more precisely, the constant matrix $\widehat{\Theta}_0$ or $\widehat{\Theta}_0^{2N}$) basically with the same idea, if we note that (11) is a special case of (16) as stated in Subsection 5.2 and that (16) can be regarded as a special case of (8) with N set to 2 and ζ replaced by z.